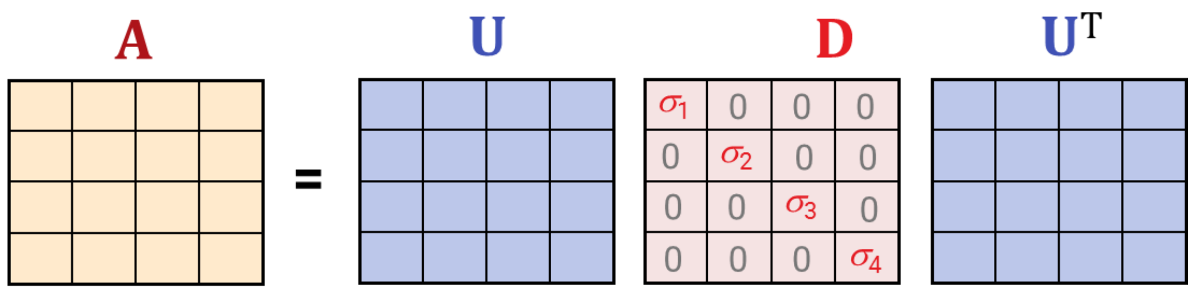
Modelling 1

SUMMER TERM 2020



LECTURE 9

Eigen- and Singular Values

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Eigenvectors and Eigenvalues

Eigenvectors & Eigenvalues

Definition:

- Linear map $\mathbf{A}$, non-zero vector $\mathbf{x}$ with

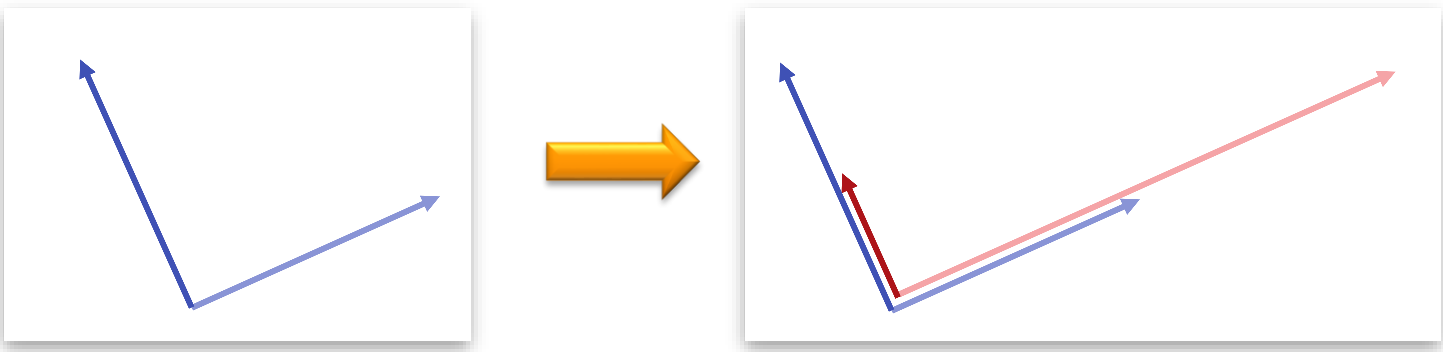
  $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

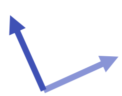
- λ an is *eigenvalue* of $\mathbf{A}$
- $\mathbf{x}$ is the corresponding *eigenvector*

Example

Intuition:

- In the direction of an eigenvector, the linear map acts like a scaling



- Example:
  - Two eigenvalues (0.5 and 2)
  - Two eigenvectors 
- Standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$: *not eigenvectors*

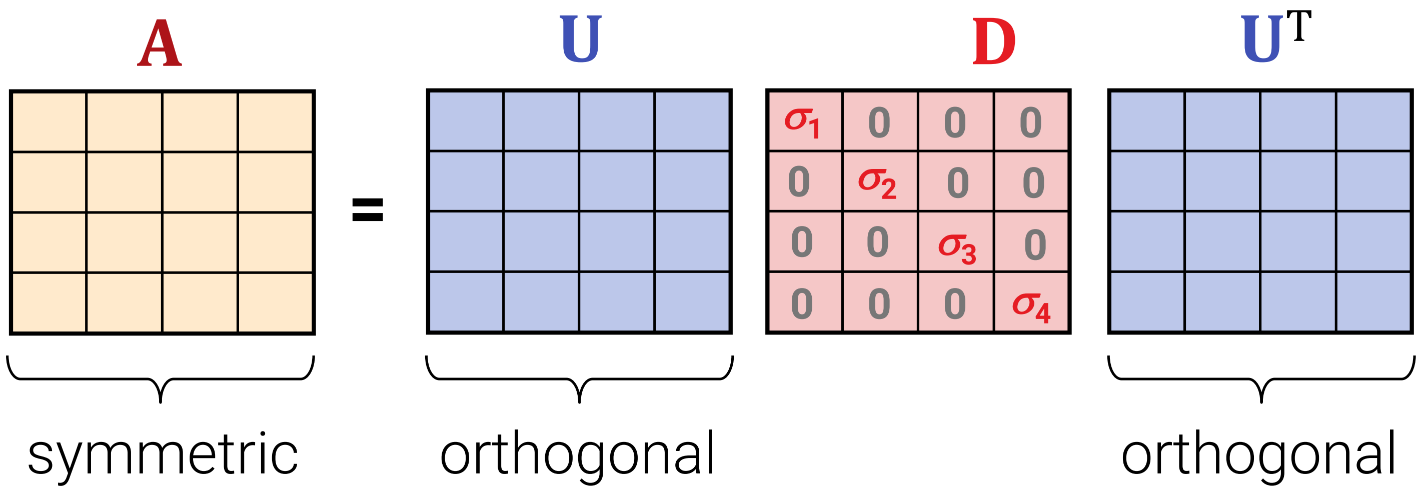
Eigenvectors & Eigenvalues

Theorem

- All real, *symmetric* matrices can be diagonalized
  - Orthogonal eigenbasis $\mathbf{U} = (\mathbf{u}_1 | \ldots | \mathbf{u}_d)$
  - $\mathbf{A} = \mathbf{U} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix} \mathbf{U}^{\mathrm{T}}$
- Symmetric matrices encode only non-uniform scaling

Diagonalization

Eigenvalue decomposition (diagonalization)



Always possible for symmetric matrices

- Symmetric: $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$

Computation

Simple algorithm

- "Power iteration" for symmetric matrices
- Computes largest eigenvalue even for large matrices
- Algorithm:
  - Start with a random vector (maybe multiple tries)
  - Repeatedly multiply with matrix
  - Normalize vector after each step
  - Repeat until ratio before / after normalization converges (this is the eigenvalue)
- Intuition:
  - Largest eigenvalue = "dominant" component/direction

Powers of Matrices

What happens:

- A symmetric matrix can be written as:

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\mathbf{U}^{\mathrm{T}}$$

- Taking it to the *k*-th power yields:

$$\mathbf{A}^k = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}}\mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}} \cdots \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}\mathbf{U}^{\mathrm{T}}$$

- EV's key to understanding powers of matrices

Generalization: SVD

Singular value decomposition:

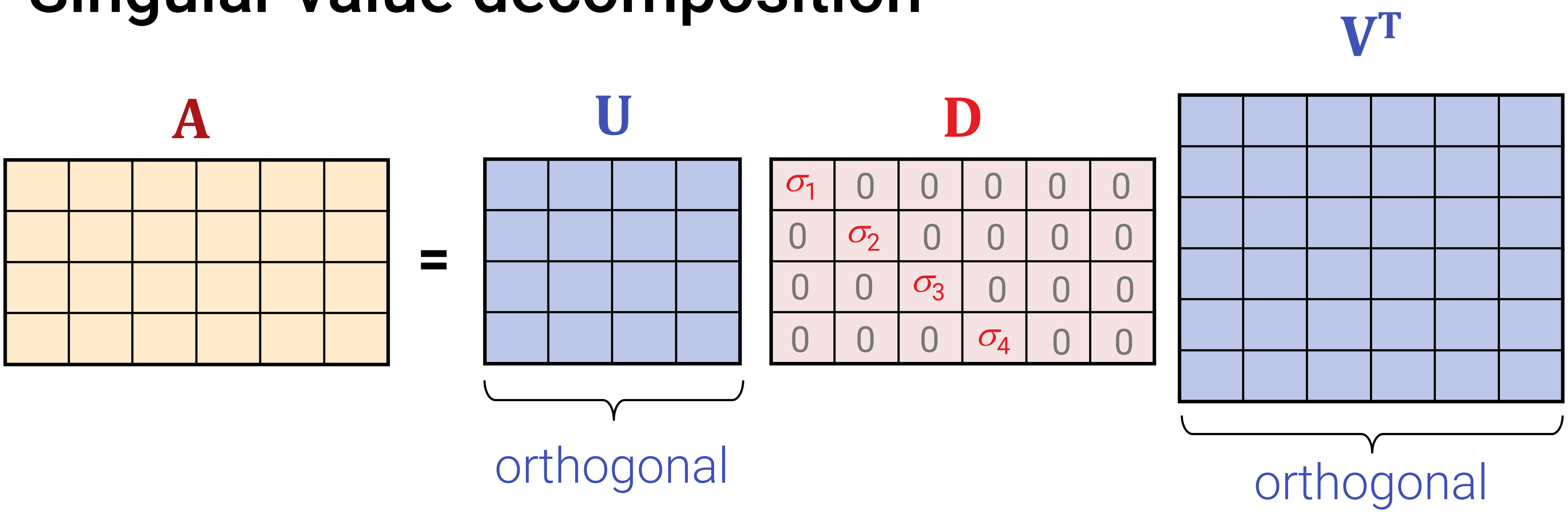
- For any real matrix $\mathbf{A}$

$$\mathbf{A} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}}$$

  - $\mathbf{U}$, $\mathbf{V}$ are orthogonal
  - $\mathbf{D}$ is a diagonal
  - Diagonal entries σ_i: "*singular values*"
- $\mathbf{U}$ and $\mathbf{V}$ are different in general
  - For symmetric matrices, they are the same
  - Then: singular values = eigenvalues
- Analogous for linear operators (∞-dim)

Singular Value Decomposition

Singular value decomposition

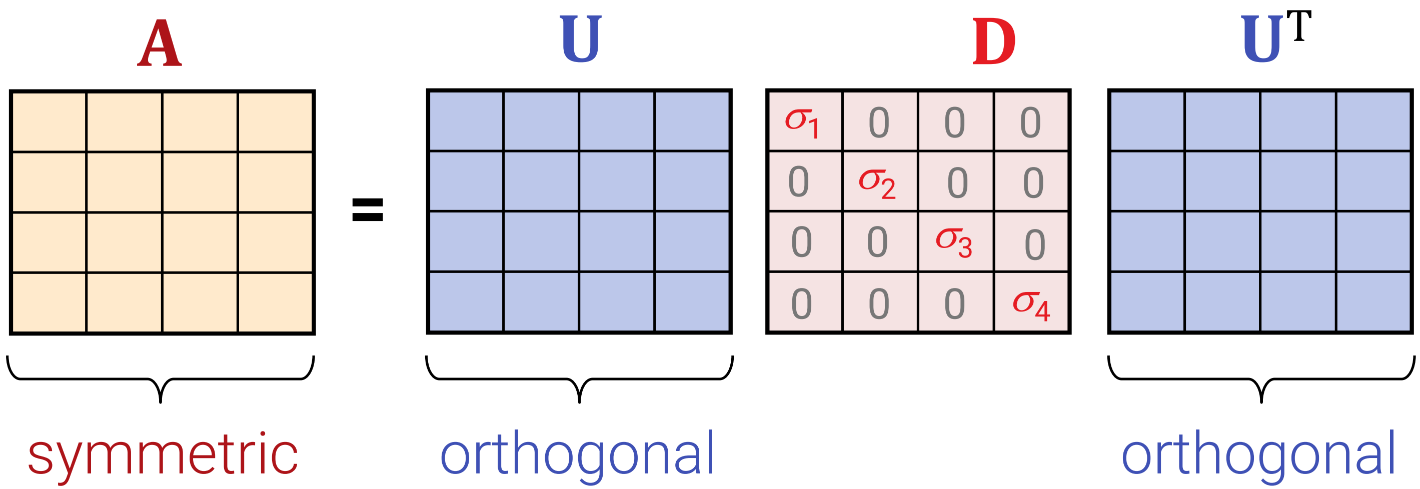


"the Swiss army knife of linear algebra"

[wikipedia user Bisco]

Comparison: Diagonalization

Eigenvalue decomposition (diagonalization)



(For symmetric matrices)

Singular Value Decomposition

SVD Solver

- For full rank, square $\mathbf{A}$:

$$\mathbf{A} = \mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}}$$

$$\Rightarrow \mathbf{A}^{-1} = (\mathbf{U}\,\mathbf{D}\,\mathbf{V}^{\mathrm{T}})^{-1} = (\mathbf{V}^{\mathrm{T}})^{-1}\,\mathbf{D}^{-1}\,(\mathbf{U}^{-1}) = \mathbf{V}\,\mathbf{D}^{-1}\,\mathbf{U}^{\mathrm{T}}$$

- Numerically very stable
- More expensive than iterative solvers
- General $\mathbf{A}$ possible (least-squares / pseudo-inverse)
  - More later